

# Liouville Field Theory on Hyperelliptic surface

S.A.Apikyan <sup>†</sup>

Theoretical Physics Departament

Yerevan Physics Institute

Alikhanyan Br.st.2, Yerevan,375036 Armenia

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## Abstract

Liouville field theory on hyperelliptic surface is considered. The partition function of the Liouville field theory on the hyperelliptic surface are expressed as a correlation function of the Liouville vertex operators on a sphere and the twist fields.

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<sup>†</sup> e-mail address: apikyan@vxc.yerphi.am

## Introduction

Last years an essential progress has been achieved in the investigations of Liouville Field Theory (LFT). Recently an analytic expression of the three-point function of the exponential fields in the Liouville field theory on a sphere is proposed [1],[2],[3]. It provides possibility for the decomposition of the multipoint correlation functions into the multipoint conformal blocks. Such a success owes much to the fact that the Liouville correlation functions with exponential operators inserted in some cases is expressed as a multipoint correlation function of vertex operators in a free field theory [4].

The present work is organized as follows. In section I a brief description is given of a Liouville Field Theory on hyperelliptical surfaces. Further by using Polyakov's proposal [5], partition function of the LFT on hyperelliptic surface reduced to LFT on sphere and free scalar field theory with inserted Liouville vertex operators and twisted fields.

### 1. LFT on hyperelliptic surfaces.

A hyperelliptic surface (HES)  $\Gamma_h$  is a compact Riemann surface of genus  $h \geq 1$  determined by an algebraic equation of the form

$$y^2(z) = P_{2h+2}(z) = \prod_{i=1}^{2h+2} (z - a_i) \quad (1.1)$$

where  $P_{2h+2}(z)$ -is a polynomial of degree  $2h + 2$ . By other words  $\Gamma_h$  is a two sheet branched covering of a Riemann sphere.

We start with Liouville field theory (LFT) in the conformal gauge on HES  $\Gamma_h$ . For the case of the non-zero genus one uses a global coordinate, provided by the Schottky uniformization [6],[7]. Local properties of the LFT are derived from the Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi}(\partial_a \phi)^2 + \mu e^{2b\phi} + \frac{Q}{4\pi} \hat{R} \phi \quad (1.2)$$

where  $b$  and  $\mu$  are coupling and cosmological constants respectively. We have fixed a background metric  $\hat{g}$  with curvature  $\hat{R}$  normalized by

$$\frac{1}{4\pi} \int \sqrt{\hat{g}} \hat{R} = 2(1 - h) \quad (1.3)$$

on a genus-h surface.

Under the hyperelliptic map (1.1) Lagrangian density  $\mathcal{L}(\phi(y))$  and conformal fields  $\phi(y), T(\phi)$  on Riemann surface  $\Gamma_h$  maps into branches  $\mathcal{L}^{(k)}(\phi^{(k)}(z)), \phi^{(k)}(z)$  and  $T^{(k)}(z)$  (i.e. branches of the original fields) ( $k = 0, 1$ ).

Each Liouville field  $\phi^{(k)}(z)$  (on each sheet separately) is not exactly a scalar field, but transforms like a logarithm of the conformal factor of metric under the holomorphic change of coordinates

$$\phi^{(k)}(\omega, \bar{\omega}) = \phi^{(k)}(z, \bar{z}) - \frac{Q}{2} \log \left| \frac{d\omega}{dz} \right|^2 \quad (1.4)$$

where

$$Q = b + \frac{1}{b} \quad (1.5)$$

On each sheet we have holomorphic Liouville stress-energy tensors

$$T^{(k)}(z) = -(\partial\phi^{(k)})^2 + Q\partial^2\phi^{(k)} \quad (1.6)$$

$$\bar{T}^{(k)}(z) = -(\bar{\partial}\phi^{(k)})^2 + Q\bar{\partial}^2\phi^{(k)} \quad (1.7)$$

with the Liouville central charge

$$\hat{c} = 1 + 6Q^2 \quad (1.8)$$

If  $A_a, B_a$  ( $a = 1, 2, \dots, h$ ) denoted the basic cycles of the HES, the monodromy operators  $\hat{\pi}_{A_a}, \hat{\pi}_{B_a}$ , form a representation of the  $Z_2$  group. We will work in the diagonal basis of the group  $Z_2$  i.e. we will choose following boundary conditions

$$T = T^{(0)} + T^{(1)}, \quad T^- = T^{(0)} - T^{(1)}, \quad \Phi = \phi^{(0)} + \phi^{(1)}, \quad \varphi = \phi^{(0)} - \phi^{(1)} \quad (1.9)$$

$$\hat{\pi}_{A_a} T = T, \quad \hat{\pi}_{A_a} T^- = T^-, \quad \hat{\pi}_{A_a} \Phi = \Phi, \quad \hat{\pi}_{A_a} \varphi = \varphi \quad (1.10)$$

$$\hat{\pi}_{B_a} T = T, \quad \hat{\pi}_{B_a} T^- = -T^-, \quad \hat{\pi}_{B_a} \Phi = \Phi, \quad \hat{\pi}_{B_a} \varphi = -\varphi \quad (1.11)$$

Under a holomorphic change of coordinates fields  $\Phi$  and  $\varphi$  transform as follows

$$\Phi(\omega, \bar{\omega}) = \Phi(z, \bar{z}) - Q \log \left| \frac{d\omega}{dz} \right|^2 \quad (1.12)$$

$$\varphi(\omega, \bar{\omega}) = \varphi(z, \bar{z}) \quad (1.13)$$

New expressions for the conserved energy-momentum tensors are given by

$$T = -\frac{1}{2}(\partial\Phi)^2 + Q\partial^2\Phi - \frac{1}{2}(\partial\varphi)^2 \quad (1.14)$$

$$T^- = -\partial\Phi\partial\varphi + Q\partial^2\varphi \quad (1.15)$$

$$\bar{T} = -\frac{1}{2}(\bar{\partial}\Phi)^2 + Q\bar{\partial}^2\Phi - \frac{1}{2}(\bar{\partial}\varphi)^2 \quad (1.16)$$

$$\bar{T}^- = -\bar{\partial}\Phi\bar{\partial}\varphi + Q\bar{\partial}^2\varphi \quad (1.17)$$

and corresponding Liouville central charge is

$$c = 2 + 12Q^2 \quad (1.18)$$

The expressions (1.14-1.18)-make obvious the splitting of the original theory into the LFT on a sphere with the central charge  $c_s = 1 + 12Q^2$  and the  $Z_2$ -orbifold theory for field  $\varphi$  with the central charge  $c_{orb} = 1$ .

We can now define  $\Phi$  as Liouville field on the whole complex plane with the following asymptotic behavior

$$\Phi(z, \bar{z}) = -2Q \log |z|^2 + reg. \quad |z| \rightarrow \infty \quad (1.19)$$

but  $\varphi$ -is usual free scalar field living on the orbifold  $S^1/Z_2$  of radius  $R$ .

According to parity of the boundary conditions (1.9-1.11) we have to define two kinds of Liouville vertex operators. First one (untwisted case) the spinless primary conformal fields

$$V_{\alpha,\beta}(x) = e^{2\alpha\Phi(x)} e^{i2\beta\varphi} \quad (1.20)$$

of dimensions

$$\Delta_{\alpha,\beta} = 2\alpha(Q - \alpha) + 2\beta^2 \quad (1.21)$$

$$\Delta_{\alpha,\beta}^- = i2\beta(Q - 2\alpha) \quad (1.22)$$

As was established in papers [10],[11] the physical LFT space of states consists of a continuum variety of primary states corresponding to operators  $e^{2\alpha\Phi}$  with

$$\alpha = ip + \frac{Q}{2} \quad (1.23)$$

and the conformal descendents of these states. Therefore the corresponding expression for the dimensions (1.21) and (1.22) looks as

$$\Delta = \frac{Q^2}{2} + 2p^2 + 2\beta^2, \quad \Delta^- = 4p\beta \quad (1.24)$$

The second one (twisted case) the primary spinless conformal fields are taken by

$$V_\gamma(x) = e^{2\gamma\Phi(x)}\sigma_{0,\epsilon}(x) \quad (1.25)$$

with dimensions

$$\Delta_{\gamma,\epsilon} = 2\gamma(Q - \gamma) + \frac{1}{16} \quad (1.26)$$

where  $\sigma_{0,\epsilon}$  ( $\epsilon = 0, 1$ ) are well-known twist (Ramond) conformal fields [8] of dimension  $\Delta_{0,\epsilon} = 1/16$ . By definition these fields  $\sigma_{0,\epsilon}$  satisfies the following nonlocal OPE's:

$$\partial\varphi(z)\sigma_{0,\epsilon}(0) = \frac{1}{2}z^{-1/2}\sigma_{1,\epsilon}(0) + \dots \quad (1.27)$$

$$\partial\varphi(z)\sigma_{1,\epsilon}(0) = \frac{1}{2}z^{-3/2}\sigma_{0,\epsilon}(0) + 2z^{-1/2}\partial\sigma_{0,\epsilon}(0) + \dots \quad (1.28)$$

where  $\sigma_{1,\epsilon}$  is second Ramond conformal field, dimension for which is defined from the general formulas of the arbitrary Ramond fields  $\sigma_{k,\epsilon}$  [9]

$$\Delta_{k,\epsilon} = \frac{(2k+1)^2}{16}, \quad k = 0, 1, 2, \dots \quad (1.29)$$

Now we have to answer how can one construct the partition function of the LFT on  $\Gamma_h$  by using vertex operators (1.20), (1.25) (with different "charges"  $\alpha, \beta, \gamma$ ). According to main proposal of Polyakov [5] the "summation" over smooth metrics with the insertion of vertex operators should be equivalent to the "summation" over metrics with singularities of the insertion points, without insertion of the vertex operators. Therefore partition function of the LFT on HES  $\Gamma_h$

$$Z_h = \int D\phi e^{-\int \frac{1}{4\pi}(\partial\phi)^2 + \mu e^{2b\phi} + Q\hat{R}^h\phi} \quad (1.30)$$

we can represente as following

$$Z_h = \int D\Phi D\varphi e^{-\int \frac{1}{8\pi}(\partial\Phi)^2 + \mu e^{2a\Phi} + Q\hat{R}^{h=0}\Phi + \frac{1}{8\pi}(\partial\varphi)^2} \prod_{i=1}^{2h+2} e^{2\gamma_i\Phi(a_i, \bar{a}_i)} \sigma_{0,\epsilon_i}(a_i, \bar{a}_i) \quad (1.31)$$

where

$$a = \frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} - \frac{1}{2}} \quad (1.32)$$

In order to evaluate the last functional integral, we first integrate over the zero mode of  $\Phi$ . Let's define following decomposition

$$\Phi(z) \equiv \tilde{\Phi}(z) + \Phi_0 \quad (1.33)$$

where  $\Phi_0$ - are the kernel of the Laplacian and  $\tilde{\Phi}(z)$ - are functions orthogonal to the kernel. After integrating over  $\Phi_0$  we read

$$\begin{aligned} Z_h = & (-\mu)^s \frac{\Gamma(-s)}{2a} \int D\tilde{\Phi} e^{-\int \frac{1}{8\pi}(\partial\tilde{\Phi})^2 + Q\hat{R}^{h=0}\tilde{\Phi}} \left( \int \sqrt{\tilde{g}} e^{2a\tilde{\Phi}} \right)^s \times \\ & \times \prod_{i=1}^{2h+2} e^{2\gamma_i \tilde{\Phi}(a_i, \bar{a}_i)} \int D\varphi \prod_{i=1}^{2h+2} \sigma_{0,\epsilon_i}(a_i, \bar{a}_i) e^{-\int \frac{1}{8\pi}(\partial\varphi)^2} \end{aligned} \quad (1.34)$$

where

$$\sum_{i=1}^{2h+2} \gamma_i = Q - sa, \quad \sum_{j=1}^{h+1} k_j = 0 \quad (1.35)$$

The second relation in (1.35) is the condition of charge's neutrality of intermediate states in the pairs OPE's [8],

$$\sigma_{0,\epsilon_1}(1)\sigma_{0,\epsilon_2}(2) = \sum z^{-1/8-2(k_{nm}^{\epsilon_1\epsilon_2})^2} e^{i2k_{nm}^{\epsilon_1\epsilon_2}\varphi(2)} \quad (1.36)$$

$$k_{nm}^{\epsilon_1\epsilon_2} = \frac{n}{R} + \frac{1}{2} \left[ m + \frac{1}{2}(\epsilon_1 + \epsilon_2) \right] R \quad (1.37)$$

where  $R$ -is radius of orbifold. Unfortunately, in general,

$$s = \frac{Q}{a} - \frac{\sum \gamma_i}{a} \quad (1.38)$$

will not be a positive integer, therefore remaining functional integral in (1.34) is not a free-field correlator. But every time when (1.35) is satisfied for integer  $s = n = 0, 1, 2, \dots$  partition function  $Z_h$  exhibits a pole in the  $\sum \gamma_i$  with the residue being specified by the corresponding perturbative integral

$$\sum_{\gamma_i=Q-na, \sum k_j=0}^{res} Z_h(a_1, \dots, a_{2h+2}) = Z_h^{(n)}(a_1, \dots, a_{2h+2}) \Big|_{\sum \gamma_i=Q-na, \sum k_j=0} \quad (1.39)$$

where  $Z_h^{(n)}$ -is free field correlator

$$Z_h^{(n)} = \frac{(-\mu)^n}{n!} \int D\Phi e^{-\int \frac{1}{8\pi}(\partial\Phi)^2 + Q\hat{R}^{h=0}\Phi} (\prod_{j=1}^n \int \sqrt{\tilde{g}} e^{2a\Phi(x_j)} d^2x_j) \times \\ \times \prod_{i=1}^{2h+2} e^{2\gamma_i\Phi(a_i, \bar{a}_i)} \int D\varphi \prod_{i=1}^{2h+2} \sigma_{0,\epsilon_i}(a_i, \bar{a}_i) e^{-\int \frac{1}{8\pi}(\partial\varphi)^2} \quad (1.40)$$

which is  $n$ -th term in the perturbative series of  $Z_h$

$$Z_h(a_1, \dots, a_{2h+2}) = \sum_{n=0}^{\infty} Z_h^{(n)}(a_1, \dots, a_{2h+2}) \quad (1.41)$$

expanded by the cosmological constant  $\mu$ . So LFT's partition function on HES reduced to Liouville correlation function (on the sphere) with inserted Liouville vertex operators (with charges  $\gamma_i$ ) and to correlation function of twisted fields  $\sigma_{0,\epsilon}$  [8]. The residue of the LFT's partition function on HES at pole  $\sum \gamma_i = Q - na$  is a correlation function of free field theory on HES.

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